APPENDIX G

Force Equilibrium: A Vectorial Approach

In this Appendix, we shall illustrate how force and moment equilibrium problems (such as Sample Problem 2.5) can be solved via a vectorial approach.

G.1 Vectors: A Review

Quantities such as positions, forces, and moments may be represented as vectors, and are denoted here by bold characters such as **p, F, M**. For example, in Figure G.1, $\mathbf{p} = 0\hat{i} + 4\hat{j} + 3\hat{k}$ is a position vector that lies on the *yz* plane, and $(\hat{i}, \hat{j}, \hat{k})$ represent the three coordinate axes; equivalently the vector $\mathbf{p} = 0\hat{i} + 4\hat{j} + 3\hat{k}$ may be captured in a tuple format as p = (0,4,3).

Given any vector $\mathbf{p} = (p_x, p_y, p_z)$, its *magnitude* is defined to be:

$$\|\mathbf{p}\| = \sqrt{p_x^2 + p_y^2 + p_z^2} \tag{G.1}$$

and its *direction* is the unit vector:

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{\|\mathbf{p}\|} \tag{G.2}$$

For example, the magnitude of vector $\mathbf{p} = (0,4,3)$ is $\|\mathbf{p}\| = \sqrt{0 + 16 + 9} = 5$ while its direction is given by $\hat{\mathbf{p}} = (0,4,3)/5$, i.e., $\hat{\mathbf{p}} = (0,0.8,0.6)$. Also observe from Equation (G.2) that, given a magnitude $\|\mathbf{p}\|$ and a direction $\hat{\mathbf{p}}$, the corresponding vector is given by:

$$\mathbf{p} = \|\mathbf{p}\|\hat{\mathbf{p}} \tag{G.3}$$

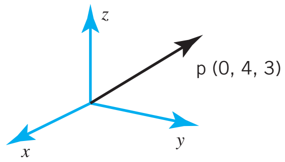

FIGURE G.1
A position vector.

For example, suppose the magnitude of a vector is 3 and its direction is the unit vector (0,–0.6, 0.8), then the corresponding vector is $\mathbf{p} = 3(0,-0.6,0.8) = (0,-1.8,2.4)$

Given any two vectors **p** and **q**, we define their *sum* as:

$$\mathbf{p} + \mathbf{q} = (p_x + q_x, p_y + q_y, p_z + q_z) \quad \textbf{(G.4)}$$

Further, their *dot-product* (resulting in a scalar quantity) is defined as:

$$\mathbf{p} \cdot \mathbf{q} = p_x q_x + p_y q_y + p_z q_z \quad \textbf{(G.5)}$$

The dot-product is particularly useful in finding the component of a vector along a given direction. For example, the component of $\mathbf{p} = (0,4,3)$ along the unit direction $\hat{\mathbf{d}} = (0,0.8,0.6)$ is given by $\mathbf{p} \cdot \hat{\mathbf{d}} = 5.0$

Finally, the *cross-product* of two vectors **p** and **q** (resulting in a vector quantity) is defined via a determinant as follows:

$$\mathbf{p} \otimes \mathbf{q} = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{bmatrix} \quad \textbf{(G.6)}$$

Evaluating the determinant, we have:

$$\mathbf{p} \otimes \mathbf{q} = (p_y q_z - q_y p_z)\hat{i} + (p_z q_x - q_z p_x)\hat{j} + (p_x q_y - q_x p_y)\hat{k} \quad \textbf{(G.7)}$$

i.e., in a tuple notation:

$$\mathbf{p} \otimes \mathbf{q} = ((p_y q_z - q_y p_z), (p_z q_x - q_z p_x), (p_x q_y - q_x p_y)) \quad \textbf{(G.8)}$$

The cross-product is particularly useful for computing moments due to external forces, as explained in the next section.

G.2 Force and Moments Equilibrium

Consider the illustration in Figure G.2*a* where an external force acts at one end of the member as shown. In the coordinate system illustrated, the force may be represented by the vector $\mathbf{F}_{ext} = F(0,0,-1) = (0,0,-F)$, where F is the magnitude of the force.

Figure G.2b illustrates a free-body-diagram taken at the cross-section *AA*. In Sample Problem 2.5, the free-body diagram was analyzed through visual inspection, and the reaction forces, moments and torque at this cross-section were determined (as shown). The objective in this Appendix is to confirm these results via a vectorial approach. The motivation behind the vectorial approach is that, for more complex problems where multiple forces are involved, visual inspection can be error-prone, whereas the vectorial approach is simple and reliable.

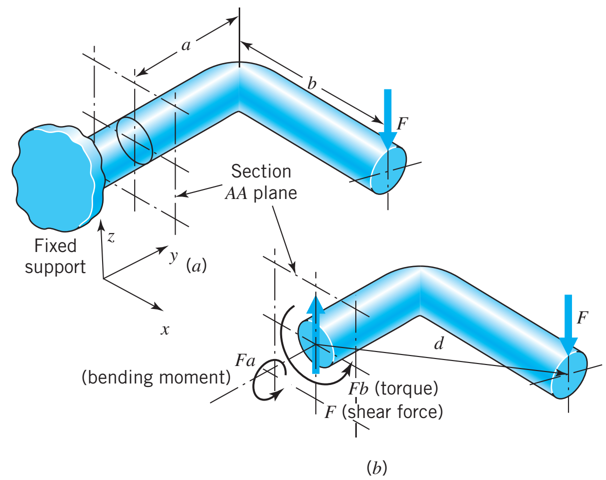

FIGURE G.2
Force acting on a member, and free-body diagram.

The vectorial approach for determining reaction forces, moments, etc. is as follows:

1. The first step is to identify all the external forces and moments. In this example, the external force is $\mathbf{F}_{ext} = (0,0,-F)$, while there are no external moments, i.e., $\mathbf{M}_{ext} = (0,0,0)$.
2. Next, we assign two unknown *internal* reaction vectors $\mathbf{F}_{int}$ and $\mathbf{M}_{int}$ at the cross-section of interest; $\mathbf{F}_{int}$ is the unknown reaction force vector, and $\mathbf{M}_{int}$ is the unknown generalized moment vector. The meaning of the generalized moment is discussed shortly. The objective is to find $\mathbf{F}_{int}$ and $\mathbf{M}_{int}$.
3. In the third step, we create a vector **d** from the center of the cross-section of interest, to the point of force application, as illustrated in Figure G.2*b*. Observe, from the dimensions marked in Figure G.2*a*, that $\mathbf{d} = b\hat{i} + a\hat{j} = (b,a,0)$.
4. Finally, we enforce the following two equilibrium equations that a free-body must satisfy:

$$\textit{Force equilibrium: } \mathbf{F}_{ext} + \mathbf{F}_{int} = \mathbf{0} \tag{G.9}$$

$$\textit{Moment equilibrium: } \mathbf{d} \otimes \mathbf{F}_{ext} + \mathbf{M}_{ext} + \mathbf{M}_{int} = \mathbf{0} \tag{G.10}$$

From the force equilibrium equation, we have:

$$\mathbf{F}_{int} = -\mathbf{F}_{ext} = -(0,0,-F) = (0,0,F) \tag{G.11}$$

In other words, the reaction force is the shear force acting in the positive *z*-direction as confirmed in Figure G.2*b*.

From the moment equilibrium equation, since there is no external moment, we have:

$$\mathbf{M}_{int} = -\mathbf{d} \otimes \mathbf{F}_{ext} \tag{G.12}$$

Since $\mathbf{F}_{ext} = (0,0,-F)$, and $\mathbf{d} = (b,a,0)$, from Equation (G.10) and Equation (G.6)

$$\mathbf{M}_{int} = -\det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ b & a & 0 \\ 0 & 0 & -F \end{bmatrix} \tag{G.13}$$

i.e., $\mathbf{M}_{int} = (aF, -bF, 0)$. In other words, the x-component of the generalized moment is aF; this is interpreted as a bending moment in Figure G.2b. The y-component of the generalized moment is $-bF$; this is interpreted as a torque in Figure G.2b (in the negative y-direction). There is no z-component of the moment.

To illustrate the generality of the vectorial approach, consider the illustration in Figure G.3, where an additional force $2F$ acts in the positive x-direction as illustrated. The objective once again is to find the internal/reaction forces $\mathbf{F}_{int}$ and moment $\mathbf{M}_{int}$ at cross-section AA.

Observe that the *external* force is now given by $\mathbf{F}_{ext} = (2F, 0, -F)$, while $\mathbf{d} = (b, a, 0)$ as before, and there are no external moments. From the force equilibrium equation, we have:

$$\mathbf{F}_{int} = -\mathbf{F}_{ext} = (-2F, 0, F) \tag{G.14}$$

i.e., there are two shear components at cross-section AA. On the other hand, the internal moment vector is given by $\mathbf{M}_{int} = -(b,a,0) \otimes (2F,0,-F)$, i.e., $\mathbf{M}_{int} = (aF, -bF, 2aF)$. In other words, there is an *additional* reaction moment of $M_z = 2aF$ at the cross-section.

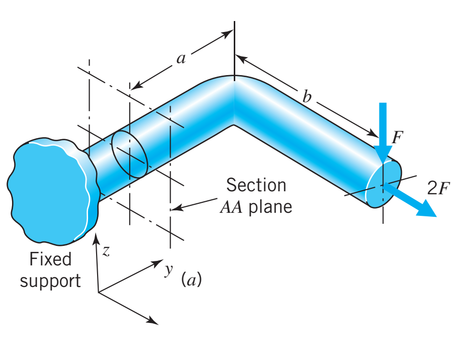

FIGURE G.3
Forces acting on a member, and free-body diagram.